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LARGE AMPLITUDE WAVES IN BOUNDED MEDIA

III. THE DEFORMATION OF AN IMPULSIVELY LOADED SLAB: THE SECOND REFLEXION

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CONTENTS

Representations are derived that describe the interactions of the waves that are reflected from both perfectly rigid and perfectly free interfaces during the arrival of a centred wave with *any* wave travelling in the opposite direction. These are used to analyse the early stages of the deformation produced when the traction at the loaded boundary continues to vary after changing discontinuously.

1. Introduction

In the first part of this paper (part II of the series) we analysed the wave interaction that occurs when a centred wave is reflected from an interface with some other material. It was shown that this problem arises during the early stages of a wide variety of technically important deformations that are produced by sudden loading. In general, only at the subsequent reflexion from the loaded boundary do the characters of these deformations begin to differ. Usually, it is at this second reflexion that the peak stresses and strains occur. In this paper we show how to analyse this second reflexion in two limiting cases: when the first reflexion is from a rigid boundary and when the first reflexion is from a perfectly free boundary.

In § 2 we derive a representation that describes the interaction of the wave reflected from a rigid interface during the arrival of a centred wave with *any* other wave travelling in the opposite direction. Since this reflected wave is also centred, this representation closely resembles that already given in part II. As a first application of this representation, in § 2.1 we analyse the interaction of this reflected wave with the wave produced by varying the applied load after it is changed discontinuously. This problem arises, for example, when a shock wave with an expansion fan behind it is incident at one of the boundaries of a slab that is held fixed at its other boundary. It also occurs when the tension at one end of a string, which is held fixed at its other end, is suddenly changed discontinuously and then allowed to vary in some prescribed manner. In particular, if the traction at the loaded boundary again changes discontinuously before the reflected wave reaches this boundary another centred wave is produced that interacts with the centred reflected wave. The interaction of these two opposite travelling centred waves is discussed in detail in § 2.2.

As another application of the representation derived in § 2, in § 3 we analyse the interaction of the centred wave reflected from the rigid boundary and the wave reflected during its arrival at the loaded boundary. This interaction is discussed for two situations: when the traction is known during this reflexion and when the velocity is known.

In § 4 we also derive a representation that describes the interaction of the wave reflected from a perfectly free interface during the arrival of a centred wave with *any* other wave travelling in the opposite direction. This is used to analyse situations that are similar to those described in §§ 2 and 3.

2. The wave reflected from a rigid interface during the arrival of a centred wave

First we derive a representation that describes the interaction of *any* β-wave with the α-wave reflected from a rigid interface during the arrival of a centred wave. Since this α-wave is also centred, the representation closely resembles that already given by equations (II, 4.8)–(II, 4.9).

Equations (II, 7.3)–(II, 7.4) imply that in the α-wave reflected from a rigid boundary

$$m(\alpha) = -(\nu+\mu\alpha)\,F'(\alpha). \tag{2.1}$$

When this expression for $m(\alpha)$ is inserted, and the second of equations (II, 9.17) is used, equation (II, 2.6) can be written

$$2A^{\frac{1}{2}}\,\partial t/\partial\alpha+[\mu(t-t_c)+\nu(X-X_c)]\,F'(\alpha) = 0. \tag{2.2}$$

Since this is of the same form as equation (II, 4.1), an argument that is almost identical with that used in II, § 4 to obtain equations (II, 4.8) and (II, 4.9) yields the equations

$$t-t_c = \psi(\beta)\,A^{-\frac{1}{2}}+\sigma(\beta) \tag{2.3}$$

and

$$X-X_c = \psi(\beta)\,A^{\frac{1}{2}}+M\sigma(\beta). \tag{2.4}$$

u can be computed from the fact that

$$u = 2G(\beta)-\hat{c}(A), \tag{2.5}$$

where $G(\beta)$ is related to $\psi(\beta)$ and $\sigma(\beta)$ by the compatibility condition

$$d\sigma/d\beta = \nu\psi\, dG/d\beta. \tag{2.6}$$

As the α-wave travels from $X = 0$ to $X = 1$ it traverses three distinct regions: the first interaction region, the simple wave region, and the second interaction region. These are labelled regions II, III and IV in figures II, 5 and II, 6. As it traverses region II, equations (II, 6.1) and (II, 6.2) imply that the variations of X and A with t at its front are given by

$$X = 1 - t^{-1} \quad \text{and} \quad A = t^{-2}. \tag{2.7}$$

When these expressions are inserted in equations (2.3) and (2.4) and then t is eliminated between these equations it follows that at the front ψ and σ are related by the equation

$$1 - \psi^2 = (\sigma + t_c)(1 - X_c - M\sigma). \tag{2.8}$$

However, since ψ and σ are constant at any β-characteristic, relation (2.8) also continues to hold throughout region II. If equations (2.3) and (2.4), which are linear in ψ and σ, are now used to express ψ and σ in terms of A, X and t, and if these expressions are inserted in the relation (2.8), an equation for $A(X, t)$ is obtained. The solution to this equation yields the expression (II, 6.9). A similar argument yields the result that $\sigma \equiv 0$ in region III. Eliminating ψ from equations (2.3) and (2.4) then yields the result (II, 9.16).

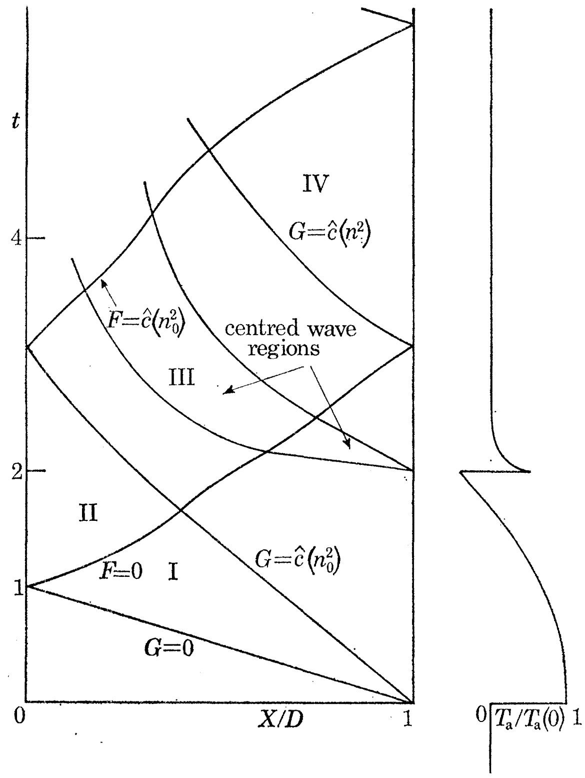

FIGURE 1. The wave pattern produced in an elastic slab which is rigidly fixed at $X = 0$ when the traction at $X = D$ varies in the manner indicated on the right. In region III′ two opposite travelling centred waves interact.

2.1. *Varying applied load*

When T varies at $X = 1$ after changing discontinuously from zero to $T_a(0)$ at $t = 0$, region III is not a simple wave region since it does not border a region where $G \equiv$ constant (see figure 1). However, if the variation in the traction at the interface does not cause a shock to form, region III does border a simple wave region in which

$$A = n^2(\beta) \quad \text{and} \quad 1 - X = n^2(\beta)\,(t-\beta). \tag{2.9}$$

$n^2(t)$ denotes the variation in A at $X = 1$ corresponding to the variation $T_a(t)$ in T. A *sufficient* condition that a shock does not form is that

$$\mathrm{d}n/\mathrm{d}\beta < \tfrac{1}{2}n^3. \tag{2.10}$$

Now the front of the reflected wave does not propagate with constant speed after crossing the region where it interacts with the incident centred wave since it is refracted by the disturbance described by equations (2.9). These equations, together with the first of equations (II, 2.1) imply that the trajectory of the front can be described by the equations

$$X = X_F(\beta) \equiv 1 - n(\beta)\left[1 - \frac{1}{2}\int_0^\beta n(s)\,\mathrm{d}s\right] \quad \text{and} \quad t = t_F(\beta) \equiv \beta + \frac{1}{n(\beta)}\left[1 - \frac{1}{2}\int_0^\beta n(s)\,\mathrm{d}s\right]. \tag{2.11}$$

In these equations β varies in the range $0 < \beta \leqslant \beta_a$, where β_a, the arrival time of the front at $X = 1$, is determined from the condition that

$$\int_0^{\beta_a} n(s)\,\mathrm{d}s = 2. \tag{2.12}$$

Note that in equations (2.11),

$$\text{as} \quad \beta \to 0+, \quad X_F \to 1 - n(0+) \quad \text{and} \quad t \to 1/n(0+). \tag{2.13}$$

In order to determine ψ and σ in region III simply insert the expressions (2.9) and (2.11) for A, t and X at the front in equations (2.3) and (2.4) and then solve these linear equations for ψ and σ. This procedure yields

$$\psi = \frac{n}{n^2 - M}[(X_F - X_c) - M(t_F - t_c)] \quad \text{and} \quad \sigma = \frac{1}{M - n^2}[(X_F - X_c) - n^2(t_F - t_c)]. \tag{2.14}$$

Now that these functions have been determined, equations (2.3)–(2.6) can be used to describe the interaction of the α-wave with the simple wave (2.9). Note that in equation (2.5)

$$G = \hat{c}(n^2). \tag{2.15}$$

This follows either from equations (2.6), (2.11) and (2.14) or, more directly, from the fact that $F \equiv 0$ and $c = \hat{c}(n^2)$ at the front (2.11).

2.2. *Interaction of two centred waves*

As an example of the general procedure described above we consider the situation when $T_a(t)$ again changes discontinuously at time $t_0 (< \beta_a)$ in such a way that *n decreases* discontinuously from $n(t_0-)$ to $n(t_0+)$. This generates a centred wave at $(X, t) = (1, t_0)$ that interacts with the centred wave reflected from $X = 0$ in some subregion (labelled region III′ in figure 1) of region III. This situation occurs, for example, when a slab that softens in compression is impacted by a blast wave that produces a pressure variation of the form shown in figure 1 at the interface

$X = 1$. It is supposed that in the time interval $0 < t < t_0$ the pressure either continues to increase or does not decrease fast enough for condition (2.10) to be violated. Alternatively, the slab could be a string which is suddenly pulled into tension at $t = 0$ and then again at $t = t_0$.

Equations (2.11) imply that as the front of the wave reflected from $X = 0$ is refracted by the wave centred at $(X, t) = (1, t_0)$

$$X_F \equiv 1 - t_1 n \quad \text{and} \quad t_F \equiv t_0 + t_1 n^{-1}, \tag{2.16}$$

where n varies in the range

$$n(t_0+) \leqslant n \leqslant n(t_0-) \tag{2.17}$$

and the constant

$$t_1 = 1 - \frac{1}{2}\int_0^{t_0} n(s)\,\mathrm{d}s. \tag{2.18}$$

When these expressions for X_F and t_F are inserted in equations (2.14) and the resulting expressions for ψ and σ are inserted in equations (2.3) and (2.4) these imply that in region III′

$$t = \psi' A^{-\frac{1}{2}} + \sigma' \quad \text{and} \quad X + 1 = \psi' A^{\frac{1}{2}} + M\sigma, \tag{2.19}$$

where

$$\psi' = (M - n^2)^{-1}[t_1 M + (Mt_0 - 2)\,n + t_1 n^2] \quad \text{and} \quad \sigma' = (M - n^2)^{-1}[2 - 2t_1 n - t_0 n^2]. \tag{2.20}$$

Equations (2.19), with ψ' and σ' given as functions of the characteristic parameter n by equations (2.20), determine the variation of A in region III′. The variation of u can then be determined from the fact that

$$u = 2\hat{c}(n^2) - \hat{c}(A). \tag{2.21}$$

Relations (2.19) and (2.20) can be used to determine A as a simple explicit function of (X, t). For, when n is eliminated from these equations a quadratic equation for A is obtained whose solution is

$$A = \frac{X^2 - Mt_0 X + M(t_0 + t_1^2) - 1}{(Mt - 2)(t - t_0) + t_1^2}. \tag{2.22}$$

Moreover, if A is considered as a function of the normalized distance and time measures

$$\bar{X} = [1 - X - s_1]/|Mt_1^2 - s_1^2|^{\frac{1}{2}} \quad \text{and} \quad \bar{t} = [M(t - t_0) - s_1]/|Mt_1^2 - s_1^2|, \tag{2.23}$$

where

$$s_1 = 1 - \tfrac{1}{2}Mt_0, \tag{2.24}$$

rather than as functions of (X, t), equation (2.22) implies that

$$A = M\bar{A}(\bar{X}, \bar{t}), \tag{2.25}$$

where

$$\bar{A} = (\bar{X}^2 + 1)/(\bar{t}^2 + 1) \quad \text{when} \quad s_1^2 < Mt_1^2 \tag{2.26}$$

while

$$\bar{A} = (\bar{X}^2 - 1)/(\bar{t}^2 - 1) \quad \text{when} \quad s_1^2 > Mt_1^2. \tag{2.27}$$

In both cases,

$$\text{at constant } \alpha, \quad \mathrm{d}\bar{X}/\mathrm{d}\bar{t} = -\bar{A} \tag{2.28}$$

and

$$\text{at constant } \beta, \quad \mathrm{d}\bar{X}/\mathrm{d}\bar{t} = \bar{A}. \tag{2.29}$$

Equation (2.28) integrates to give

$$\bar{X} = \begin{cases} (\bar{\alpha} - \bar{t})/(1 + \bar{\alpha}\bar{t}) & \text{when (2.26) holds;} \\ (\bar{\alpha} - \bar{t})/(1 - \bar{\alpha}\bar{t}) & \text{when (2.27) holds:} \end{cases} \tag{2.30}$$

equation (2.29) integrates to give

$$\bar{X} = \begin{cases} (\bar{t} - \bar{\beta})/(1 + \bar{\beta}\bar{t}) & \text{when (2.26) holds;} \\ (\bar{t} - \bar{\beta})/(1 - \bar{\beta}\bar{t}) & \text{when (2.27) holds.} \end{cases} \tag{2.31}$$

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In equations (2.30) and (2.31), $\bar{\alpha}$ and $\bar{\beta}$ are characteristic parameters. To determine $\bar{\beta}$ as a function of n use the fact that according to equations (2.16) and (2.23) at the front of the α-wave

$$\bar{X} = (t_1 n - s_1)/|Mt_1^2 - s_1^2|^{\frac{1}{2}} \quad \text{and} \quad \bar{t} = (Mt_1 n^{-1} - s_1)/|Mt_1^2 - s_1^2|^{\frac{1}{2}}. \tag{2.32}$$

When these expressions are inserted in (2.30) and (2.31) these equations yield the result that

$$\bar{\beta} = \bar{\alpha}_F \frac{n^2 - M}{n^2 - 2M(t_1/s_1)\,n + M}, \tag{2.33}$$

where the value of the characteristic parameter $\bar{\alpha}$ at the front

$$\bar{\alpha}_F = \begin{cases} s_1^{-1}(Mt_1^2 - s_1^2)^{\frac{1}{2}} & \text{when (2.26) holds,} \\ -s_1^{-1}(s_1^2 - Mt_1^2)^{\frac{1}{2}} & \text{when (2.27) holds.} \end{cases}$$

Equations (2.33) and (2.31) can be used to determine n as an explicit function of (X, t). Condition (2.21), with A determined as an explicit function of $(\bar{X}, \bar{t})$ from equations (2.25)–(2.27), then determines u as an explicit function of $(\bar{X}, \bar{t})$.

3. The second reflexion

When the α-wave that is reflected from the rigid interface $X = 0$ during the arrival of a centred wave reaches $X = 1$ at $t = \beta_a$ it starts to interact with the wave reflected from this interface. The form of this reflected wave is determined by what happens at $X = 1$. The simplest situation is when $T_a(t)$, and consequently $n(\beta)$, is known during this second reflexion. Then, in region IV (see figure 1) the deformation is still described by equations (2.3)–(2.6). Now though, in the expressions (2.14) for ψ and σ

$$X_F \equiv 1 \quad \text{and} \quad t_F = \beta. \tag{3.1}$$

3.1. *Reflexion from an interface at which T is constant*

In the special case when T is constant, $= \tau$ say, at $X = 1$ during the second reflexion, so that n is constant in region IV, equations (2.3), (2.4), (2.14) and (3.1) imply that

$$A = \tfrac{1}{4}n^2[(1+\eta) + [(1+\eta)^2 - 4Mn^{-2}\eta]^{\frac{1}{2}}]^2, \quad \text{where} \quad \eta = (1-X)/(Mt-2). \tag{3.2}$$

These equations determine A as an explicit function of (X, t) in region IV. Then, $\beta(X, t)$ can be determined from the fact that

$$\beta = [(M-n^2)\,tA^{\frac{1}{2}} + 2(n - A^{\frac{1}{2}})]/n[M - nA^{\frac{1}{2}}], \tag{3.3}$$

although a much simpler relation that can be used to calculate the β = constant curves is

$$t - \beta = (M\beta - 2)\,(1-X)/[(M\beta-2)\,n^2 + (n^2 - M)\,(1-X)]. \tag{3.4}$$

To determine $u(X, t)$, we must first calculate $G(\beta)$ from condition (2.6). This is easily done for when the expression (2.14) for ψ and σ are inserted, with X_F and t_F given by equations (3.1), equation (2.6) implies that

$$(2\nu + \mu\beta)\,\mathrm{d}G/\mathrm{d}\beta = n. \tag{3.5}$$

This equation integrates to give

$$G = \hat{c}(n^2) + \mu^{-1} n \ln\left(\frac{M\beta - 2}{M\beta_a - 2}\right). \tag{3.6}$$

$u(X,t)$ can now be computed from conditions (2.5), (3.2), (3.3) and (3.6). The α-characteristics can be determined directly from the fact that

$$\text{at constant } \alpha, \quad \mathrm{d}X/\mathrm{d}t = -A, \tag{3.7}$$

where A is given by equation (3.3). Equation (3.7) integrates to give the result that at constant α,

$$(Mt-2)\frac{A^{\frac{1}{2}}}{M-nA^{\frac{1}{2}}}\exp[\mu n^{-\frac{1}{2}}\hat{c}(n^2)-\hat{c}(A)] = \frac{n^{\frac{1}{2}}}{M-n^2}(M\bar{\alpha}-2), \tag{3.8}$$

where $\bar{\alpha}$ denotes the arrival time of the characteristic at $X = 1$.

3.2. *Shock formation*

The main effect of the wave reflected from an interface at which T is constant is to harden the material in region IV. An argument that is almost identical to that used in II, § 6.2 shows that this hardening is sufficient to produce a shock if A attains the value

$$A_s = n^{-2}M^2[1-(1-M^{-1}n^2)^{\frac{1}{2}}]^2. \tag{3.9}$$

Clearly, this cannot occur for a non-ideal material because then $n^2 \geqslant M$. However, for a non-ideal material it always occurs if the original centred wave is strong enough to produce yield during the first reflexion from $X = 0$ (see II, figure 21). If yield does not occur at $X = 0$ a shock can still form in region IV if

$$A_L \leqslant A_s \leqslant n^2, \tag{3.10}$$

where A_L is the least value of A in region IV. A_L can be determined in terms of M, n and the strength of the original centred wave. To do this note that according to equation (3.2) A decreases monotonically with t at the front of the wave reflected from $X = 1$ (see figure 1) so that $A = A_L$ as the front crosses the back of the α-wave reflected from $X = 0$. There,

$$c = c_L = \hat{c}(n_0^2)+\hat{c}(n^2), \quad \text{where} \quad n_0^2 = A_a(0+) \quad \text{and} \quad A_L = \hat{A}(c_L). \tag{3.11}$$

When the appropriate expressions for $\hat{A}(c)$ are used, conditions (3.10) and (3.11) imply that a shock will only form when

$$\frac{M(n_0+n-1)-n_0 n}{M+n_0 n-(n_0+n)} < \frac{M}{n}[1-(1-M^{-1}n^2)^{\frac{1}{2}}]. \tag{3.12}$$

When condition (3.12) is satisfied, an argument that is identical to the one used in II, § 6.2 shows that the shock forms at the front of the wave reflected from $X = 1$ at the point (X_{s_4}, t_{s_4}), where

$$X_{s_4} = X_c + 2\left(\frac{M-n}{M-n^2}\right)A_s^{\frac{1}{2}} \quad \text{and} \quad t_{s_4} = t_c + 2\left(\frac{M-n}{M-n^2}\right)A_s^{-\frac{1}{2}}. \tag{3.13}$$

Condition (3.12) on M, n and n_0 can be replaced by an equivalent condition on M, τ/T_1 and $T_a(0)/T_1$. Figure 2 depicts the relation between M and the least value of $T_a(0)/T_1$ that will cause a shock to form when $\tau = T_a(0)$ $(n = n_0)$ and when $\tau = 0$ $(n = 1)$. The first case occurs when T is held constant at $X = 1$ after changing discontinuously at $t = 0$. The second case occurs when the load at $X = 1$ has again returned to its ambient value by the time the wave that is reflected from $X = 0$ reaches $X = 1$. Note that in this later case condition (3.12) implies that a shock will form only if

$$A_a(0) = n_0^2 < M^2[1-(1-M^{-1})^{\frac{1}{2}}]^2. \tag{3.14}$$

However, according to (II, 7.15) the right hand side is the greatest value of $A_a(0)$, which corresponds to the least value of $T_a(0)/T_l$, that will cause the material to yield at $X = 0$. Consequently if the material does not yield at $X = 0$ a shock cannot form in region IV when $n = 1$. When $\tau/T_l < 0$ $(n > 1)$ a shock can only form in region IV when the material yields at $X = 0$. It should be noted that the above results are only valid if a shock has not already formed in region III.

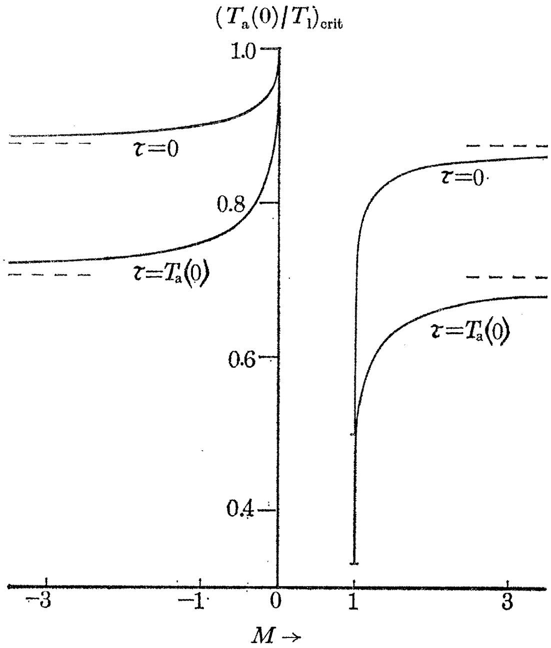

FIGURE 2. The relation between M and the least value of $T_a(0)/T_l$ that will cause a shock to form in region IV when $T = T_a(0)$ and when $T = 0$ at $X = 1$ during the second reflexion.

3.3. *Reflexion from an interface at which u is constant*

When u, rather than T, is specified at $X = 1$, $n(\beta)$ must be calculated before equations (2.3)–(3.14) can be used to describe the deformation in regions III and IV. This is easily done in region III, where $0 < \beta \leqslant \beta_a$. For, since $F = 0$ at $X = 1$ when $0 < t \leqslant \beta_a$,

$$n^2 = \hat{A}(u_a(\beta)) \quad \text{for} \quad 0 < \beta \leqslant \beta_a, \tag{3.15}$$

where $u_a(t)$ denotes the variation of u at $X = 1$. In region IV, though, $n(\beta)$ is a little more difficult to calculate. To do so use the fact that according to equation (2.5)

$$G(\beta) = \tfrac{1}{2}[u_a(\beta) + \hat{c}(n^2)]. \tag{3.16}$$

When this expression for G is inserted in (2.6), with ψ and σ given by (2.14) with $X_F = 1$ and $t_F = \beta$, the equation

$$\frac{1}{n^2 - M}\frac{dn}{d\beta} + \frac{n}{2 - M\beta} = \tfrac{1}{2}\nu\frac{du_a}{d\beta} \tag{3.17}$$

is obtained for $n(\beta)$. In general, this equation must be integrated numerically. However, in the special case when u_a is constant, $= v$ say, it integrates to give

$$n^2 = M[1 + K(M\beta - 2)^2]^{-1}, \tag{3.18}$$

where the constant

$$K = \frac{M - \hat{A}(v)}{\hat{A}(v)\,(2 - M\beta_a)^2}. \tag{3.19}$$

When the expression (3.18) for $n(\beta)$ is inserted in equations (2.14) these yield simple expressions for $\psi(\beta)$ and $\sigma(\beta)$ in region IV. When these expressions are inserted in equations (2.3) and (2.4) the resulting equations can be solved to give

$$A = M\frac{1+K(1-X)^2}{1+K(Mt-2)^2} \tag{3.20}$$

and

$$M\beta-2 = \frac{Mt+X-3}{1+K(1-X)(Mt-2)}. \tag{3.21}$$

Now that A is known as an explicit function of (X, t) the α-characteristic can easily be determined. They are given by the condition that at constant α

$$\frac{Mt-X-1}{1-K(1-X)(Mt-2)} = \text{constant}, \quad = M\bar{\alpha}-2 \quad \text{say}. \tag{3.22}$$

u can be calculated from equations (2.5) and (3.16) which imply that

$$u = v+\hat{c}(n^2)-\hat{c}(A), \tag{3.23}$$

where A is given by (3.20) and n^2 is determined as an explicit function of (X, t) from conditions (3.18) and (3.21).

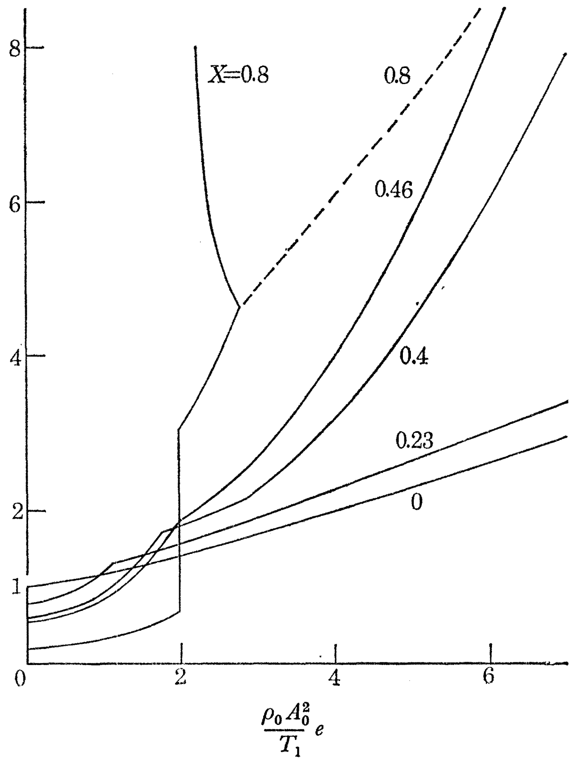

FIGURE 3. Typical variations of strains at several representative particles when $X = 0$ is rigid and when T is held constant at $X = 1$. The value of $M = 1.1$ and $T_a = 0.94T_1$.

3.4. *Illustration*

Figure 3 depicts typical variations of strain at several representative particles for the case when $X = 0$ is a rigid boundary and when T is held constant at $X = 1$ up to and during the second reflexion. The characteristic net associated with this deformation is shown in figure II, 6. The value of $M = 1.1$ and $T_a = 0.94T_1$. The particle $X = 0.23$ lies in region I for some time, and then in region II at all subsequent times. The particle $X = 0.40$ lies in region I and then in region II for finite time intervals, and then in region III at all subsequent times. The particle $X = 0.46$ lies in region I for a finite time, then in region III at all subsequent times. Finally, the particle $X = 0.8$ lies in region I for a finite time, then in the region where $T = T_a$ for a finite interval of time, and then in the second interaction region IV until it is influenced by the wave that is reflected from the shock (see figure II, 6).

The broken curve in figure 3 depicts the variation in strain at $X = 0.8$ when u, rather than T, is held fixed at $X = 1$ up to and during the second reflexion. Over the time interval considered the deformation at the other particles is the same as when T is held constant.

4. The wave reflected from a perfectly free interface during the arrival of a centred wave

In this section we obtain a representation that describes the interaction of the wave reflected from a perfectly free interface during the arrival of a centred wave with *any* other wave travelling in the opposite direction. To do this note that according to equations (II, 6.5) and (II, 6.6), in this reflected wave

$$m \equiv 1 \quad \text{and} \quad F = \mu^{-1} \ln \frac{1-M\alpha}{1-M}. \tag{4.1}$$

When this information is inserted, equation (II, 2.6) and the second of equations (II,2.1) imply that variations of X and t with c at constant β are governed by the linear equations

$$2A^{\frac{1}{2}}\,\partial t/\partial c + [\mu t + \nu(1+X)] = 2(1-M^{-1})\exp[\mu(c-G)] \quad \text{and} \quad \partial X/\partial c = -A\,\partial t/\partial c. \tag{4.2}$$

These integrate to give

$$t - M^{-1} = \psi(\beta)\,A^{-\frac{1}{2}} + \sigma(\beta) + (A^{-\frac{1}{2}} - M^{-1})\exp[\mu(\hat{c}(A) - G)], \tag{4.3}$$

and

$$X = \psi(\beta)\,A^{\frac{1}{2}} + M\sigma(\beta) + (1 - A^{\frac{1}{2}})\exp[\mu(\hat{c}(A) - G)]. \tag{4.4}$$

Equations (4.3) and (4.4) are only compatible with the first of equations (II, 2.1) if ψ, σ and G satisfy the compatibility condition

$$\mathrm{d}\sigma/\mathrm{d}\beta = \nu\psi\,\mathrm{d}G/\mathrm{d}\beta. \tag{4.5}$$

Once the functions ψ, σ and G are known the variation of A with (X, t) follows from equations (4.3) and (4.4), while the variation of u follows from (2.5). For example, in region II

$$\psi \equiv 0, \quad \sigma \equiv 0 \quad \text{while} \quad 0 \leqslant G \leqslant c_a. \tag{4.6}$$

Also, if T is held constant at $X = 1$ after changing discontinuously at $t = 0$, in region III

$$\sigma \equiv 0, \quad G \equiv c_a \quad \text{and} \quad 0 \leqslant \psi \leqslant 1. \tag{4.7}$$

4.1. *Varying applied load*

The calculation of ψ, σ and G in region III when T varies at $X = 1$ is straightforward. G is again given by (2.15) while, according to equations (4.3) and (4.4), ψ and σ are determined from the linear equations

$$t_F - n^{-1} = n^{-1}\psi + \sigma \quad \text{and} \quad X_F - 1 + n = n\psi + M\sigma, \tag{4.8}$$

where $X_F(\beta)$ and $t_F(\beta)$ are given in terms of $n(\beta)$ by equations (2.11). These conditions yield

$$G = \hat{c}(n^2), \quad \psi = \frac{n}{M-n^2} M\beta + \frac{1}{2}\frac{n^2+M}{n^2+M}\int_0^\beta n(s)\,\mathrm{d}s \quad \text{and} \quad \sigma = \frac{n^2}{n^2-M}\beta + \frac{n}{M-n_2}\int_0^\beta n(s)\,\mathrm{d}s. \tag{4.9}$$

In particular, in region III′ where the wave reflected from $X = 0$ interacts with a wave centred at $(X, t) = (1, t_0)$ equations (4.9) imply that

$$G = \hat{c}(n^2), \quad \psi = Mt_0 \frac{n}{M-n^2} + (1-t_1)\frac{n^2+M}{n^2-M} \quad \text{and} \quad \sigma = t_0 \frac{n^2}{n^2-M} + 2(1-t_1)\frac{n}{M-n^2}. \tag{4.10}$$

When the expressions (4.10) are inserted in equations (4.3) and (4.4) these provide explicit expressions for X and t as functions of A and the characteristic parameter n. Unfortunately, these equations cannot be used to determine A as an explicit function of (X, t) as was the case when $X = 0$ was a perfectly rigid interface. However, the trajectories of constant T and e can easily be obtained by holding A at its approximate constant value and varying n over the range (2.17) in these expressions.

4.2. *Reflexion from an interface at which T is constant*

When T is constant, $= \sigma$, during the reflexion from $X = 1$ it is a simple matter to calculate ψ, σ and G in region IV. In terms of the characteristic parameter

$$\lambda = \exp[\mu(d-G)], \tag{4.11}$$

$$\psi = \frac{n-1}{n+1}\lambda + K\lambda^{-1/n} \quad \text{and} \quad \sigma = M^{-1}\left[1 + \frac{n-1}{n+1}\lambda - nK\lambda^{1/n}\right], \tag{4.12}$$

where the constants d and n are the values of c and $A^{\frac{1}{2}}$ corresponding to $T = \sigma$, and the constant

$$K = \frac{n}{M-n^2}\left(M\beta_a - 2\frac{M+n}{1+n}\right). \tag{4.13}$$

With $\psi(\lambda)$ and $\sigma(\lambda)$ given by equations (4.12), in region IV equations (4.3) and (4.4) can be written

$$t - M^{-1} = \psi A^{-\frac{1}{2}} + \sigma + \lambda(A^{-\frac{1}{2}} - M^{-1})\exp[\mu(\hat{c}(A) - c_a)] \tag{4.14}$$

and

$$X = \psi A^{\frac{1}{2}} + M\sigma + \lambda(1 - A^{\frac{1}{2}})\exp[\mu(\hat{c}(A) - c_a)]. \tag{4.15}$$

Equations (2.5) and (4.11) imply that

$$u = 2d - c - 2\mu^{-1}\ln\lambda. \tag{4.16}$$

In particular, at $X = 1$, where $c \equiv d$, equations (4.14) and (4.16) imply that

$$u = d - 2\mu^{-1}\ln\lambda, \tag{4.17}$$

where the variation of λ with t follows from the relation

$$t = 2M^{-1} + 2\frac{1-M^{-1}}{1+n}\lambda + K\frac{M-n^2}{Mn}\lambda^{1/n}. \tag{4.18}$$

4.3. *Reflexion from an interface at which u is constant*

When u remains constant at $X = 1$ during the second reflexion, it is best to express G, ψ and σ as function of the characteristic parameter n in region IV. Although the procedure for calculating these functions is straightforward, the algebra involved is rather messy: only the result will be quoted.

When $u = v$ at $X = 1$,

$$G = \tfrac{1}{2}(\hat{c}(n^2)+v), \quad \psi = \exp\left[\tfrac{1}{2}\mu(\hat{c}(n^2)-v)\right] - K|n^2-M|^{-\frac{1}{2}} \tag{4.19}$$

and

$$M\sigma = 1-\exp\left[\tfrac{1}{2}\mu(\hat{c}(n^2)-v)\right] + Kn|n^2-M|^{-\frac{1}{2}}, \tag{4.20}$$

where the constant

$$K = M|M/\hat{A}(v)-1|. \tag{4.21}$$

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